ROBUSTLY TRANSITIVE ACTIONS OF $\mathbb{R}^2$ ON COMPACT THREE MANIFOLDS

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ABSTRACT. In this paper, we define C^1-robust transitivity for actions of $\mathbb{R}^2$ on closed connected orientable manifolds. We prove that if the ambient manifold is three dimensional and the dense orbit of a robustly transitive action is not planar, then it is "degenerate" and the action is defined by an Anosov flow.

1. INTRODUCTION

In some recent works in the theory of dynamical systems robust transitivity of diffeomorphisms and flows has been investigated. Weak forms of hyperbolicity has been shown to be necessary conditions for robust transitivity of flows and diffeomorphisms of compact manifolds. Bonatti-Díaz-Pujals [1] proved that C^1-robustly transitive diffeomorphisms admits dominated splittings. Previous to this work, Díaz-Pujals-Ures [2] had proved that robustly transitive diffeomorphism on three dimensional manifolds are partiall hyperbolic. For C^1-flows, there are also results which imply that some weak form of hyperbolicity is necessary to obtain robust transitivity. See for example a result of Vivier [11] about robustly transitive flows on any dimension and a result of Doering [3] in three dimensional case.

By using Kupka-Smale theorem in one dimensional case, one deduce that there does not exist any robustly transitive diffeomorphism on one dimensional manifolds. Also, by a result of Peixoto [5], we know that the Morse-Smale flows form a dense subset of the set of C^1-flows on any surface. This result implies that robustly transitive flows may exist only on manifolds with dimension higher than two.

If we consider the diffeomorphisms or flows defined on a manifold as the action of $\mathbb{Z}$, $\mathbb{R}$ on it, a naturally question arises: "what about robustly transitive actions of higher dimensional groups?"

In this paper, we begin the study of robustly transitive actions of $\mathbb{R}^2$ by giving some examples of these actions and proving that in three dimensional manifolds the only robustly transitive actions of $\mathbb{R}^2$ (We do not consider the case when all orbits are planar) are defined by robustly transitive flows (see theorem 1.2). On the other hand we know that by a result of Doering, robustly transitive flows are Anosov flows.

Let N denote a closed connected orientable three manifold and $\varphi : \mathbb{R}^2 \times N \to N$ be a C^r-action. For each $w \in \mathbb{R}^2 \setminus \{0\}$, φ induces a C^r-flow $(\varphi_w^t)_{t \in \mathbb{R}}$ given by $\varphi_w^t(p) = \varphi(tw, p)$ and its corresponding C^{r-1}-vector field X_w is defined by $X_w(p) = D_1\varphi(0, p) \cdot w$. If $\{w_1, w_2\}$ is a base of $\mathbb{R}^2$, the associated vector fields X_{w_1}, X_{w_2} satisfy the commutativity condition $[X_{w_1}, X_{w_2}] = 0$ and determine completely the action φ. They are called *infinitesimal generators* of φ. This condition of commutativity between two vector fields is a necessary and sufficient condition for them to be generators of an action. $X_{(1,0)}$ and $X_{(0,1)}$ are called the *canonical infinitesimal generators*.

Denote by $A^r(\mathbb{R}^2, N)$ $1 \le r \le \omega$ the set of actions of $\mathbb{R}^2$ on N whose infinitesimal generators are of class C^r. Given two actions $\{\varphi; X_{(1,0)}, X_{(0,1)}\}$ and $\{\psi; Y_{(1,0)}, Y_{(0,1)}\}$ define

$$d_{(1,1)}(\varphi, \psi) = \max\{\|X_{(1,0)} - Y_{(1,0)}\|_1, \|X_{(0,1)} - Y_{(0,1)}\|_1\}.$$

With this distance $A^r(\mathbb{R}^2, N)$ is a metric space and its corresponding topology is called the $C^{(1,1)}$-*topology*. Note that this topology is finer than the C^2-topology and coarser than the C^1-topology. For any action $\phi \in A^r(\mathbb{R}^2, M)$, $\mathcal{O}_p := \{\phi(\omega, p), \omega \in \mathbb{R}^2\}$ is called the orbit of $p \in M$. The orbit is called *singular* if its dimension is less than two.

Definition 1.1. An action $\varphi \in A^1(\mathbb{R}^2, N)$ is called *transitive* if it admits a dense orbit in N. φ is *robustly transitive* if it admits a $C^{(1,1)}$ neighborhood $\mathcal{U}$ of transitive actions.

Our main result is the following theorem.

Theorem 1.2 (Main Theorem). *Let N be a closed orientable 3-manifold. Assume that $\varphi \in A^2(\mathbb{R}^2, N)$ is robustly transitive with a dense orbit which is not homeomorphic to $\mathbb{R}^2$. Then, φ is defined by an Anosov flow.*

We mention that the hypotheses about the topological type of the dense orbit is important to our result. By this hypotheses, the dense orbit is cylindrical or homeomorphic to $\mathbb{R}$. However, we conjecture that the same result is true without this hypotheses.

We would like to thank C. Bonatti for mentioning us that, Rosenberg had left the stability

problem of the action with all leaves planar (which is the case we are avoiding here) as an open problem.

To prove the above theorem, we study the topological type of the obits of a robustly transitive action. In general one can have three different topological type for non singular (two dimensional) orbits of an action of $\mathbb{R}^2$. But in the context of transitive action, we show that the topological types are more restricted. We show that for the purpose of proving our theorem, we can suppose that the action has a dense cylindrical orbit. Then, we apply a closing lemma for actions (proves by Roussarie for locally free actions) and prove that $C^{(1,1)}$-close to a robustly transitive action, one can find an action with many compact leaves. The existence of sufficiently large amount of these tori will contradict the existence of a dense leaf and consequently robust transitivity of the initial action.

2. Examples and basic Results

Let us give some examples of robustly transitive actions of $\mathbb{R}^2$.

Example 2.1. Firstly we construct a singular (defined by flow) example of a robustly transitive action. Consider a robustly transitive Anosov vector field X defined in $\mathbb{T}^3$ and let $\phi \in A(\mathbb{R}^2, \mathbb{T}^3)$ be the action defined by $X_1 := X$ and $X_2 := cX (c \in \mathbb{R})$.

It is obvious that $[X_1, X_2] = 0$ and so they define an action of $\mathbb{R}^2$ in $\mathbb{T}^3$. Clearly, all orbits of this action are singular. We claim that ϕ is robustly transitive. Indeed, suppose $\psi \in A(\mathbb{R}^2, \mathbb{T}^3)$ any $C^{(1,1)}$ perturbation of ϕ. By the definition of $C^{(1,1)}-$topology in $A(\mathbb{R}^2, \mathbb{T}^3)$ we conclude that $\tilde{\phi}$ is defined by $\tilde{X}_1, \tilde{X}_2$ such that $[\tilde{X}_1, \tilde{X}_2] = 0$ and $\tilde{X}_1$ is C^1- close to X_1. So, $\tilde{X}_2$ is also an Anosov flow. By a result of Kato-Morimoto [4] the centralizer of an Anosov vector field is trivial. This means that $\tilde{X}_2 = f\tilde{X}_1$ (f is a first integral.) and consequently ψ is also defined by a transitive flow. On the other hand, let Y be any flow C^1-close to X, then Y and cY defines an action which is $C^{(1,1)}-$close to ϕ and consequently transitive. This implies that Y is a transitive flow and so, X is C^1-robustly transitive. So, we showed that ϕ is defined by a robustly transitive flow.

Let denote by X^t the flow of a vector field X.

Example 2.2. Let N be a three manifold supporting a robustly transitive Anosov flow. In the second example, we construct a robustly transitive action in $M^4 = N \times \mathbb{S}^1$ which is not defined by a flow. Consider the coordinate system (x, θ) in $M^4, x \in N, \theta \in \mathbb{S}^1$. In what follows, for a real function $a(x, \theta)$, by $a(x, \theta)\frac{\partial}{\partial x}$ we mean $a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + a_3\frac{\partial}{\partial x_3}$ where x_1, x_2, x_3 are coordinates in N.

Let $\phi \in A(\mathbb{R}^2, M^4)$ be defined by X_1 and X_2 such that $X_1 = a(x, \theta)\frac{\partial}{\partial x}$ is a robustly transitive Anosov flow in N and $X_2 := \frac{\partial}{\partial \theta}$. We claim that ϕ is robustly transitive.

Consider a $C^{(1,1)}$-perturbation ψ of the initial action ϕ. It is generated by two vector fields Y_1 and Y_2 which are respectively C^1 close to X_1 and X_2.

Let $N_0 := \{(x,0) : x \in N\}$. By transversality of X_2 to N_0 and closedness of X_2 and Y_2 we conclude that Y_2 is also transverse to N_0.

In our coordinate systems

$$Y_1 = \tilde{a}(x,\theta)\frac{\partial}{\partial x} + b(x,\theta)\frac{\partial}{\partial\theta}.$$

$$\tilde{X}_2 = c(x,\theta)\frac{\partial}{\partial x} + d(x,\theta)\frac{\partial}{\partial\theta}.$$

where b and c are close to zero in C^1-topology, a and $\tilde{a}$ are close in each coordinates and d is close to constant 1. We define

$$\Pi(Y_1) = dY_1 - bY_2 = (\tilde{a}d - bc)\frac{\partial}{\partial x}.$$

Observe that $\Pi(Y_1)$ is a C^1-vector field close to X_1 and consequently it is transitive. The intersection of the orbits of ψ with N_0 coincide with the orbits of $\Pi(Y_1)$. Let $x_0 \in N_0$ with a $\Pi(Y_1)$ dense orbit. We claim that the orbit of ψ passing through $(x_0, 0)$ is dense in M^4. To see this, just observe that N_0 is a global transverse manifold for Y_2. Let $U \in M^4$ an open set and $V = \bigcup_{t\in\mathbb{R}} Y_2^t(U) \cap N_0$. Then, V is an open subset of N_0 and by transitivity of $\Pi(\tilde{X}_1)$ there exists $x \in N_0$ such that for some $t \in \mathbb{R}$, $Y_1^t(x) \in V$ and consequently for some $s \in \mathbb{R}, Y_2^s(Y_1^t(x)) \in U$ and this finishes the proof.

Here we outline some basic results about the action of $\mathbb{R}^2$ which will be used in our proof of the main result. Recall that, for any action $\phi \in A^1(\mathbb{R}^2, M)$, $\mathcal{O}_p := \{\phi(\omega, p), \omega \in \mathbb{R}^2\}$ is the orbit of $p \in M$ and $G_p := \{\omega \in \mathbb{R}^2 : \phi(\omega, p) = p\}$ is called the isotropy group of p. One of the simple results is the following.

Lemma 2.3. *Suppose that $\mathcal{O}(q)$ is accumulated by $\mathcal{O}(p)$ then $G_p \subseteq G_q$.*

Proof. To prove, just observe that for $\omega \in G_p$, by definition of action and isotropy group we have $\phi(\omega, \phi(\eta, p)) = \phi(\eta, p)$. So, by continuity of ϕ we conclude that, if z is an accumulation point of $\mathcal{O}(p)$ then $\phi(\omega, z) = z$ and consequently we have $\omega \in G_q$. $\qquad\square$

Using the above lemma we can show that all the dense orbits of φ have the same topological type. In the setting of Theorem 1.2 all the dense orbits are either line or cylinder. In fact we point out that the existence of a dense line prohibits the existence of any (not necessarily dense) cylinder. This is just by the continuity of action. Consequently, if the dense orbit for a transitive action is homeomorphic to $\mathbb{R}$ then it is given by a transitive flow.

Claim 2.4. *Any two dense orbits are homeomorphic.*

Proof. To prove the claim let $\mathcal{O}_p, \mathcal{O}_q$ two dense orbits with isotropy groups G_p, G_q. The density of $\mathcal{O}_p$ implies that $G_p \subset G_q$. Similarly the density of $\mathcal{O}_q$ implies that $G_q \subset G_p$ and consequently $G_p = G_q$ and the orbits are homeomorphic. □

Lemma 2.5. *If $\phi \in A^1(\mathbb{R}^2, M)$ has a dense cylinder orbit, then any two dimensional orbit is either torus or cylinder.*

Proof. Suppose that $\mathcal{O}_p$ is a dense cylinder orbit. The isotropy group of p is $\mathbb{Z}u$ for some $0 \neq u \in \mathbb{R}^2$. Let Y be the vector field whose flow corresponding is $Y^t = \phi(tu, \cdot)$. It is clear that $Y^1(p) = p$. Let X be the vector field whose flow corresponding is $X^t = \phi(tv, \cdot)$, where v is any linearly independent to u. Then X and Y commute and consequently every point on $\mathcal{O}_p$ is periodic with period one for Y. Now, using denseness of $\mathcal{O}_p$ we conclude that any point of the manifold is a periodic point for Y which finishes the proof of the lemma. □

3. Closing lemma and proof of the main result

First of all let us recall the closing lemma of Pugh ([6, Theorem 6.1]) for the flows in a two dimensional manifold.

Theorem 3.1. *Let $X \in \mathcal{X}(M^2)$ have a nontrivial recurrent tranjectory through $p^* \in M$, let U be a neighbourhood of p^* and $\epsilon > 0$ be given. Then, there exists $Z \in \mathcal{X}(M)$ such that:*

  (1) *$X - Z$ vanishes om $M \setminus U$,*
  (2) *the C^1-size of $X - Z$ is less than ϵ respecting the U-coordinates,*
  (3) *Z has a closed orbit through p^*.*

In [9], Roussarie and Weil proved a closing lemma for the action of $\mathbb{R}^2$ on three manifolds. More precisely one of their results is the following:

Theorem 3.2. *Let N be an orientable compact closed C^r ($r > 2$), 3-manifold and φ a locally free C^r-action. If all orbits of φ are not planar, then there is a locally free action $\varphi_1 \in A^r(\mathbb{R}^2, N)$ with a compact orbit and C^1-close to φ.*

To prove the above theorem, the authors firstly observe that either φ has a compact orbit or all the orbits are dense. In the latter case just take $\varphi = \varphi_1$. In the former case, the denseness of all orbits is a corollary of a result of Sacksteder [10] about the minimal sets of $\mathbb{R}^{n-1}$ actions on $n-$manifolds. The result of Saksteder states that there is no exceptional minimal set for locally free actions. Using the denseness of a cylinder, one can show that all other orbits are cylindrical. In this setting (all the orbits are cylindrical) the proof of Pugh

closing lemma for flows on surfaces can be carried on to prove that φ can be perturbed to give a compact orbit.

Let us mention that the above theorem is not the main result of Roussarie and Weil´s paper. In fact, their paper is mainly dedicated to the proof of the following theorem [9, Theorem 2 (1)].

Theorem 3.3. *Let ϕ be a C^r-action. For all non-planar and recurrent orbit Λ e for all $\epsilon > 0$ there exists a submanifold diffeomorphic to $\mathbb{T}^2$, ϵ-close to Λ such that the plane field tangent to this submanifold can be extended to a plane field C^1 near to plane field correponding to ϕ.*

The main issue in this result is to find a nearby torus to the recurrent leaf. Here we have a general action which can have singularities. However, we suppose that there exists a dense cylinder. The idea is using again the closing lemma of Pugh.

Theorem 3.4. *Let N be an orientable compact closed, 3-manifold and $\varphi \in A^1(\mathbb{R}^2, N)$. If there exists a dense orbit $\mathcal{L}$ of φ homeomorphic to $\mathbb{S}^1 \times \mathbb{R}$, then there is an action $\varphi_1 \in A^1(\mathbb{R}^2, N)$ with a compact orbit and $C^{(1,1)}-$close to φ. Moreover, the perturbation is supported on a neighbourhood of a closed orbit of Y where Y is one of the infinitesimal generating vector fields of φ.*

To use the closing lemma of Pugh, we should take care for some technical problems which arises when we are dealing with actions. In fact the idea is as following: Whenever, we have a dense cylinder we choose a closed orbit of one of the infintesimal generating vector fields and take an adequate system of coordinates around this closed orbit. An small box in our new coordinates will serve as a flow box of the closing lemma of Pugh. All the orbits passing through this box are two dimensional. Like in the closing lemma for flows, we have a transversal section, which is a ring in our case. We construct this ring foliated by closed orbits of Y. Now, we should take care about the returns of the dense orbit in our neighbourhood. More precisely, in the following lemma we show that the dense orbit returns and intersects the transversal section in closed orbits. So, we really can carry the proof of the closing lemma for flows to our case.

The following lemma was announced in [8] for non-singular actions. Here we show that it is true for general actions. In the following lemma X and Y are two generating infinitesimal vector fields for $\phi \in A^1(\mathbb{R}^2, M)$.

Lemma 3.5. *Let $\mathcal{O}_p$ be a dense cylindrical orbit of $\phi \in A^1(\mathbb{R}^2, M)$ and c (homeomorphic to S^1) be the periodic orbit of Y passing through p. Then, for any neighbourhood $U(c)$ of c there exist an unbounded sequence $t_i \in \mathbb{R}$ such that $X^{t_i}(c) \subset U(c)$.*

Proof. Let $\mathcal{U}_\epsilon$ be an $\epsilon-$neighbourhood of c in M such that $Y^t(z) \in U(c)$ for any $z \in \mathcal{U}, t \in [0, 1]$. By density of $\mathcal{O}_p$ there exist $z \in c, t \in \mathbb{R}$ such that $X^t(z) \in \mathcal{U}_\epsilon$. It comes out that $Y^I(X^t(z)) \in U(c)$ where $Y^I(.)$ stands for $\{Y^s(.), s \in I = [0, 1]\}$. But by commutativity

$$Y^I(X^t(z)) = X^t(Y^I(z)) = X^t(c) \in U(c).$$

As ϵ can be any small number, we conclude that there is a sequence $t_i \to \infty$ such that $X^{t_i}(c) \in U(c)$. □

Infinitesimal generators adapted to a $S^1 \times \mathbb{R}$-orbit. Let $\mathcal{O}_p$ be a cylindrical orbit of $\varphi \in A^1(\mathbb{R}^2, N)$ and $\{w_1, w_2\}$ be a set of generators of its isotropy group G_p. Write $X = X_{w_1}, Y = X_{w_2}$ and Y has periodic orbit of period one through p. Note that if $q \in \mathcal{O}_p$, then the orbit of Y passing through q is periodic of period one too. Put a Riemannian metric on N and let ξ be the norm one vector field defined in a neighborhood of the Y^I_p that is orthogonal to the orbits of φ.

Let c be the circle orbit of Y through p. We know that all the orbits close to c are cylindrical or torus. For small $\epsilon > 0$ we define the ring $A_\epsilon = \{Y^I(\xi^t(c)), |t| \leq \epsilon\}$. As the action is φ is orientable and ϵ is small, A_ϵ is diffeomorphic to $S^1 \times (-\epsilon, \epsilon)$. We parametrize c with $\theta \in [0, 1]$ such that $\frac{\partial}{\partial \theta} = Y|c$. We put a coordinate system (x, θ, z) in a small neighbourhood of c diffeomorphic to $S^1 \times (-\epsilon, \epsilon) \times (-1, 1)$ such that:

$$X = \frac{\partial}{\partial x}, Y = \frac{\partial}{\partial \theta}, \xi = \frac{\partial}{\partial z}.$$

In this new coordinates system the (pieces of) orbits of φ inside such neighbourhood are $z = $ constant.

3.1. **Proof of the main theorem.** Let $\mathcal{U}$ be a $C^{(1,1)}$ neighborhood of φ such that every action in $\mathcal{U}$ is transitive.

We will prove that φ is defined by a flow. It is obvious (by continuity) that if the dense orbit of a transitive action is one-dimensional then the action is defined by a flow (The two infinesimal vestor fieds are linearly dependent.) In what follows we will show that a robustly transitive action can not have a dense cylinder. So, we conclude that in fact φ is a robustly transitive flow and by a result of Doering [3], it comes out that φ is an Anosov flow.

Suppose that φ has a dense cylinder $\mathcal{O}_p$. Let c be the periodic orbit (homeomorphic to S^1) through p and A_ϵ the ring defined previously. Recall that $\{z = 0\} \cap A_\epsilon = c$ and all $\{z = t\} \cap A_\epsilon, |t| \leq \epsilon$ are periodic orbits of the generating vector fireld Y. By lemma 2.5 all

two dimensional orbits are either cylindrical or homeomophic to torus. Before continuing the argument we state an algebraic topological lemma. [1]

Lemma 3.6. *Let N be a three dimensional compact orientable manifold. There exists $k \in \mathbb{N}$ such that if $T_1, T_2, \cdots, T_k$ are submanifolds homeomorphic to torus $\mathbb{T}^2$, then they form the boundary of a three dimensional submanifold of N.*

Having in mind the above lemma, we conclude that if A has k compact orbits then there can not exist any dense orbit. Indeed, any dense two dimensional submanifold should intersect one of these k tori.

So, let $\mathcal{O}_p$ be a dense orbit and for small ϵ all the orbits passing through A_ϵ be cylindrical. In fact, if there does not exist such an ϵ we conclude that there are more than k torus and using the above lemma we contradict the denseness of $\mathcal{O}_p$. For $0 \le i \le k-1$ Let

$$A_i := \{\frac{\epsilon i}{k} < z < \frac{(i+1)\epsilon}{k}\}.$$

By the denseness of $\mathcal{O}_p$ and lemma 3.5 there exists a return time $\bar{t}$ such that $X^{\bar{t}}(c) \in \{|z| < \frac{\epsilon}{k}\}$. As $X^{\bar{t}}(p) \in \{|z| < \frac{\epsilon}{k}\}$ we project $X^{\bar{t}}(p)$ along the orbit of X and find out t such that $X^t(p) \in A_0$. By definition, A_ϵ is foliated by the orbits of Y and by the commutativity of X and Y one concludes that $X^t(c) \in A_0$ for some t. Indeed,

$$X^t(Y^s(p)) = Y^s(X^t(p)) \in A_0 \quad \text{for all} \quad s \in [0,1]$$

which means that $X^t(c) \in A_0$.

Now, we use the closing lemma and perturb φ inside $\{|z| < \frac{\epsilon}{k}\}$ and find a new action φ_1 and $C^{(1,1)}$-close to φ with a compact orbit. As φ_1 is also transitive, it has a dense orbit which we claim it is of cylindrical type. To see this remember that our perturbation is supported on $\{|z| < \frac{\epsilon}{k}\}$ and consequently the orbits passing through $A_i, i > 0$ remains cylindrical. So, the dense orbit of φ_1 which necessarily intersect $\{\frac{1}{k} < z < \frac{2}{k}\}$ is cylindrical. Pertubing again by the closing lemma we obtain another invariant torus and by induction we find $\varphi_k \in A^2(\mathbb{R}^2, N)$ with k compact leaves which by lemma 3.6 form the frontier of a compact three manifold with boundary inside N and so, no dense orbit can exist which gives a contradiction.

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